A New Window Function for Signal Spectrum Analysis and FIR Filter Design

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Abstract: A new window function is presented which like the well known Hamming window offers a preferred property for use in signal spectrum analysis: the sum of window coefficients with its shifted version by half of the order (50% overlap) is constant for the overlapped region in the time domain. In high orders, the new window has main-lobe width equal to Hamming window, while featuring 2–4 dB smaller maximum side-lobe peak. For low orders, the window parameters are modified to have smaller main-lobe width compared to Hamming window, while maintaining smaller maximum side-lobe peak. Our results indicate performance improvement of the proposed window compared to Kaiser and Gaussian windows. A comparison with Dolph-Chebyshev window is also presented. The FIR filters designed by windowing method show the efficiency of the new window.

Keywords: window, FIR, Hamming, Kaiser, Dolph-Chebyshev.

I. INTRODUCTION

Two main applications of the windows in digital signal processing are: data analysis based on Fast Fourier Transform (FFT) and design of Finite Impulse Response (FIR) filters from Infinite Impulse Response (IIR) filters. For FFT analysis, windows are employed to suppress the so-called “leakage effect”, and for FIR filter design according to the “windowing method”, Gibbs oscillations are attenuated [1]. Desirable characteristics for a window in the frequency domain are small main-lobe width and side-lobe peak (high attenuation). However, these two requirements are contradictory, since for a given length, a window with a narrow main-lobe has a poor attenuation, and vice versa. Also a third preferred property of window when applied in data spectrum analysis is that, in the time domain, the sum of window function \( w[n] \) with its shifted version by \( M/2 \) samples \((M = \text{the window order})\) would be constant:

\[
w[n] + w[n - M/2] = \text{constant} \quad M/2 \leq n \leq M \quad (1)
\]

For example, in analyzing the non-stationary signals (such as speech processing), the signal is partitioned into several overlapped frames to yield approximately stationary properties; these frames are, in fact, the windowed versions of the original signals, and if the applied window satisfies property (1), the cost of computations is considerably decreased. The rectangular, Bartlett, Hann, and Hamming windows offer this advantage, but other windows such as Blackman, Kaiser, Gaussian, and Dolph-Chebyshev [1-3] do not satisfy property of (1).

In the recent decade, many researches have focused on designing new windows to improve the performance in particular aspects, having their own advantages and drawbacks. Unispherical windows [4], which are derived from Gegenbauer polynomials, have the property that the side-lobes roll-off can be controlled by a parameter, independent from the main-lobe width and first side-lobe level; therefore all classes of windows can be approximated by this window. Data windows given by the inverse of the product of two Gamma functions [5], offer the same advantage of controlling side-lobes decay, but in trade-off with the main-lobe width. Another class of windows based on Gegenbauer polynomials introduced in [6], beside of having controllable tradeoff between main-lobe width and side-lobes peak, also have an additional parameter to combine the following two criteria in designing FIR filters by desired ratio: minimizing side-lobes peak, and/or minimizing their energy. This class of windows allows analytical expressions in both of time and frequency domains. A family of windows based on Legendre polynomials has also been reported in [7]. The windows in [4-7] have relatively high computational complexity in deriving the windows coefficients; therefore they may not be used in real-time applications. Furthermore, they do not have property of (1). Two windows based on the “Sinc” function have been introduced in [8] and [9], which offer smaller side-lobes peak compared to Hamming and Blackman windows, respectively, while having the same main-lobe widths as them; but, they do not have property (1).

In this paper, we present a new window function which can be considered as a special case of the important class of windows, named raised cosine windows [10]. The proposed window has 2–4 dB more side-lobe attenuation than that of Hamming window, while offering approximately the same main-lobe width, and still satisfying the property in (1). The window parameters are modified to avoid the performance degradation for lower window lengths, which happens for Hamming window.

The rest of the paper is organized as follows: Section II describes the main idea in obtaining the new window, which is based on a modification to the Hamming window. In Section III, we compare the frequency response of the proposed window with the Hamming, Kaiser, Gaussian, and Dolph-Chebyshev windows. In order to evaluate the performance of the proposed window, FIR low-pass filter design by windowing method, as an application example, is given in Section IV. Finally, Section V concludes the paper.

II. PROPOSED WINDOW

Our goal is to modify Hamming window to lower its maximum side-lobe peak, while holding the main-lobe width unchanged, and still satisfying property of (1). In the following, we explain the derivation of the new window. All windows described later, have zero valued coefficients outside the interval \( 0 \leq n \leq M \).

Hamming window has the shape of:

\[
w_{h}[n] = 0.54 - 0.46 \cos(2\pi n / M) \quad 0 \leq n \leq M \quad (2)
\]
This window satisfies the property mentioned in (1), i.e.
\[ w_a[n] + w_a[n - M/2] = 0.54 + 0.54 \quad M/2 \leq n \leq M \] (3)

On the other hand, Blackman window is composed of three terms as:
\[ w_b[n] = 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M) \]
\[ 0 \leq n \leq M \]

that is, it has a DC term, a cosine function with frequency 2\pi/M, and its second harmonic. For this window, due to the second harmonic, the property in (1) is not satisfied:
\[ w_a[n] + w_a[n - M/2] = 0.84 + 0.16\cos(4\pi n/M) \]
\[ M/2 \leq n \leq M \]

\[ w_a[n] + w_a[n - M/2] = 2a_0 \]
\[ M/2 \leq n \leq M \] (5)

However, it can be found that if the third harmonic is added to the Hamming window function, then the property of (1) will be satisfied. Therefore, the main idea in obtaining the new window is to insert a third harmonic of cosine function into (2). Thus, we obtain an extra degree of freedom in tuning the window coefficients. In this way, the proposed window will be as:
\[ w[n] = a_0 - a_1 \cos(2\pi n/M) - a_3 \cos(6\pi n/M) \]
\[ 0 \leq n \leq M \] (6)

Where for normalization, i.e. \( w[M/2] = 1 \), we have:
\[ a_0 + a_1 + a_3 = 1 \] (7)

The new window is also symmetric about point M/2; thus it has a generalized linear phase, like the other common windows. Checking for the property in (1), we find that:
\[ w[n] + w[n - M/2] = 2a_0 \]
\[ M/2 \leq n \leq M \] (8)

Another point of view states that eq. (6) is a four-term raised cosine window [10], with restriction that the third term is zero:
\[ w[n] = \sum_{i=0}^{K} a_i \cos \left( \frac{2\pi i n}{M} \right) \]
\[ 0 \leq n \leq M, \quad K = 3, \quad a_2 = 0 \] (9)

The new window can be analyzed in the frequency domain. Its Fourier transform is:
\[ W(\omega) = \left[ a_0 D(\omega) + \frac{a_1}{2} \left[ D(\omega - \frac{2\pi}{M}) + D(\omega + \frac{2\pi}{M}) \right] \right] + \frac{a_3}{2} \left[ D(\omega - \frac{6\pi}{M}) + D(\omega + \frac{6\pi}{M}) \right] \exp \left( -j\frac{M\omega}{2} \right) \] (10)

where \( D(\omega) \) is Dirichlet kernel [10]:
\[ D(\omega) = \frac{\sin((M+1)\omega/2)}{\sin(\omega/2)} \] (11)

The phase shift in Eq. (10) is due to shifting the window in time domain by M/2 samples. Fig. 1 depicts Five kernels in \( W(\omega) \) (ignoring the phase shift term) for typical values of \( a_0 = 0.5, \ a_1 = 0.4, \ a_3 = 0.1, \) and \( M=12 \). We observe that all five kernels are zero at \( \omega = 4\pi/M \); therefore, like Hamming window, the main-lobe width for this window is 8\pi/M.

Furthermore, it is seen that at frequencies \pm 6\pi/M, the only nonzero terms are the kernels \( D(\omega-6\pi/M) \) or \( D(\omega+6\pi/M) \). Since the goal is to achieve a maximum side-lobe peak smaller than that of Hamming window, which is -42.6 dB, one limit on the new window parameters will be as:
\[ 20\log \left( \frac{a_3/2}{a_0} \right) < -42.6 \text{ dB} \] (12)

Noting the above condition and the normalizing condition in (7), we can apply a simple optimization algorithm to find the optimal values of the window parameters. For sufficiently large orders, the derived window is of the form:
\[ w[n] = 0.536 - 0.461\cos(2\pi n/M) - 0.003\cos(6\pi n/M) \]
\[ 0 \leq n \leq M \] (13)

Just like Hamming window, the frequency response of the new window is degraded for low orders; therefore depending on the window order, the above parameters are modified to maintain the efficiency. Figure 2 shows the dependence of \( a_0 \) and \( a_1 \) on \( M \). It can be easily verified that the coefficients are composed of a monotonic function and a DC term. We tried some simple formulas to present the dependence of these parameters on \( M \); the following formulas approximately fit the data obtained from the optimization:
\[ a_0 = 0.537 - \frac{0.3}{M+15} ; \quad a_1 = 0.46 + \frac{0.25}{M+15} ; \quad a_3 = 1 - a_0 - a_1 \] (14)

The above coefficients have relatively simple forms and can be easily calculated.
Fig. 3 compares the shape of the proposed and Hamming windows for a typical window order of $M=40$. The proposed window has lower values than Hamming window for all samples; the differences are more obvious at the two ends.

III. PERFORMANCE EVALUATION

In this section, we compare frequency domain characteristics of the proposed window with some commonly used windows.

A. Hamming Window

Hamming window has been widely used in different applications because of its good specifications. Figure 4 compares Fourier transform of the Hamming and proposed windows for different values of $M$. Figure 4.a shows that, for low window orders ($M=10$), the proposed window offers both smaller main-lobe width and side-lobes peak compared to Hamming window.

As $M$ increases, the main-lobe width of the two windows become approximately equal, as we can see from Figs. 4.b and 4.c for $M=40$ and $M=200$, respectively; but the maximum side-lobes peak for the new window is smaller.

Data presented in Table I indicate that, the maximum side-lobe peak for the proposed window is $2-4$ dB smaller than that of Hamming window, while having equal (for sufficiently high orders) or smaller (for low orders) main-lobe width. Since the Fourier transform is symmetric about $\omega=0$, only half of the main-lobes are shown in Fig. 4, and the actual values are twice, as mentioned in Table I.

Since Hamming window has more side-lobe attenuation than Bartlett and Hanning windows, we conclude that our window also outperforms Bartlett and Hanning windows, of course in the case of maximum side-lobes peak.

B. Kaiser Window

Kaiser window has the following shape:

$$w_K[n] = I_0\left(\sqrt{1 - \left(\frac{2n}{M}-1\right)^2} \beta\right) \quad 0 \leq n \leq M$$

where $\beta$ is the tuning parameter of the window to obtain the desired “main-lobe width – side-lobe peak” tradeoff, and $I_0(.)$ is the modified Bessel function of order zero.

Figures 5.a and 5.b demonstrate the performance of the proposed and Kaiser windows for $M=40$ and two different values of $\beta$. For $\beta=6.07$, Fig. 5.a shows that the Kaiser window has equal maximum side-lobe peak compared to the proposed window ($-44.4$ dB), while the proposed window has less main-lobe width ($2 \times 0.102$ compared to $2 \times 0.109$). If we want the Kaiser window to have the same main lobe width of the new window, then we can increase its length to $M+1=44$. Therefore the proposed window offers the desired specifications with lower length. However, if we need the same window length and main lobe width of the proposed window, then we can decrease $\beta$. Fig. 5.b demonstrates that for $\beta=5.55$, the main lobe widths of the two windows are the same, but the proposed window has less side-lobe peak ($-44.4$ dB compared to $-41.1$ dB). Note that the Kaiser window does not satisfy the property in (1).

![Figure 3. Proposed and Hamming windows in the time domain, $M=40$](image)

![Figure 4. Fourier transform of the proposed and Hamming windows for a) $M=10$, b) $M=40$, c) $M=200$; note on different intervals on frequency axis](image)

<table>
<thead>
<tr>
<th>$M$</th>
<th>Proposed window</th>
<th>Hamming window</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>main lobe width</td>
<td>side lobe peak</td>
</tr>
<tr>
<td>10</td>
<td>$2 \times 0.425$</td>
<td>$-40.6$ dB</td>
</tr>
<tr>
<td>40</td>
<td>$2 \times 0.102$</td>
<td>$-44.4$ dB</td>
</tr>
<tr>
<td>200</td>
<td>$2 \times 0.02$</td>
<td>$-44.2$ dB</td>
</tr>
</tbody>
</table>

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Figure 5. Fourier transforms of the proposed and Kaiser windows for \( M = 40 \); shaping parameter for Kaiser window is (a) \( \beta = 6.07 \), (b) \( \beta = 5.55 \).

C. Gaussian Window

The Gaussian window is of the form:

\[
    w_G[n] = \exp\left(-\frac{1}{2} \left(\frac{n-M/2}{\sigma M/2}\right)^2\right) \quad 0 \leq n \leq M
\]  

where \( \sigma < 0.5 \) is the tuning parameter of the window. For \( \sigma = 0.445 \), Fig. 6 shows that the proposed window offers 8 dB less side lobe peak (-44.4 dB compared to -36.4 dB).

D. Dolph-Chebyshev (D-Ch) Window

This window can be expressed as a cosine series [4] of:

\[
    w_D[n] = \frac{T_M(x_0) + \sum_{i=1}^{M/2} T_M \left( x_0 \cos \left( \frac{i\pi}{M+1} \right) \right) \cos \left( \frac{2n\pi}{M+1} \right)}{M+1} \quad 0 \leq n \leq M
\]  

where \( T_M(x) \) is the Chebyshev polynomial of degree \( M \), and \( x_0 \) is a function of the side lobe ratio. It has high cost of computation, but the important property is that all its side-lobes are equal, and the main-lobe width is the minimum that can be achieved for a given ripple ratio. As we can see from Fig. 7, this window gives -48.2 dB peak of side lobe (3.8 dB better ripple ratio compared to the proposed window); but the proposed window coefficients can be computed easier, and furthermore, Dolph-Chebyshev window does not satisfy property in (1). Also as we will see in the next section, the FIR filters designed by two windows have very close attenuations, and in some cases, the new window results in even better ripple ratios.

**IV. APPLICATION EXAMPLE**

In order to evaluate the performance of proposed window, and compare the results with the other windows, an example of FIR low pass filter design by windowing method is considered. Having a cut-off frequency of \( \omega_c \), the impulse response of an ideal low pass filter is:

\[
    h_{LP,ideal}[n] = \frac{\sin(\omega_c n)}{\pi n}
\]  

By windowing this IIR filter with the windows mentioned before, different FIR filters are obtained. Figure 8 depicts frequency responses of the FIR filters obtained by applying five different windows of length \( M+1=201 \) (for clarity, only the stop band of the filters are shown).

In fact, depending on \( M \), the filter designed by the new window offers lower or higher ripple ratios compared to the Dolph-Chebyshev windows, but it is always better than the Hamming, Kaiser, and Gaussian windows. For \( M = 200 \), the proposed window is better than all mentioned windows, as we observe from Fig. 8. For \( M = 100 \) the same results hold true. On the other hand, for \( M = 70 \) and \( M = 40 \), Dolph-Chebyshev window gives the smallest ripple ratio, but the difference with that of the new window is small. Table II summarizes the results of this experiment. However, note that the Kaiser and Dolph-Chebyshev windows have the disadvantage of high cost of computation in calculating the window coefficients, while the proposed window has a simple closed form. Furthermore, as mentioned before, the Kaiser and Dolph-Chebyshev windows do not satisfy the property in (1).
TABLE II. ATTENUATIONS (dB) OF THE FIR FILTERS OBTAINED BY WINDOWING OF AN IDEAL LOW PASS FILTER WITH DIFFERENT WINDOWS.

<table>
<thead>
<tr>
<th></th>
<th>$M=40$</th>
<th>$M=70$</th>
<th>$M=100$</th>
<th>$M=200$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>-55</td>
<td>-55.6</td>
<td>-53.1</td>
<td>-53.5</td>
</tr>
<tr>
<td>Hamming</td>
<td>-52.3</td>
<td>-52.4</td>
<td>-54.9</td>
<td>-53.6</td>
</tr>
<tr>
<td>Dolph-Chebyshev</td>
<td>-59.5</td>
<td>-61.4</td>
<td>-57.6</td>
<td>-59.3</td>
</tr>
<tr>
<td>Proposed</td>
<td>-59.8</td>
<td>-61.1</td>
<td>-58.3</td>
<td>-60.6</td>
</tr>
</tbody>
</table>

V. CONCLUSION

A novel efficient window function having a closed simple formula and the property of (1) was presented. The new window has the main lobe width less than or equal to that of the Hamming window while offering less maximum side-lobe peak. The performance comparison of the proposed window with the other windows showed the better performance of the proposed window. The average reduction in the side lobe peak of the new window compared to that of the Hamming, Kaiser, and Gaussian windows is 3 dB, 3.3 dB, and 8 dB, respectively. The FIR filter designed with the proposed window achieves less ripple ratio than those of obtained using the above windows for all window lengths, and also compared to Dolph-Chebyshev window.

REFERENCES


Figure 8. Frequency responses of FIR low pass filters ($M=200$) obtained by windowing of an IIR ideal low pass filter ($\omega_c =0.25\pi$) with the proposed window and a) Hamming b) Kaiser, c) Gaussian d) Dolph-Chebyshev windows.