Locking Time of Oscillators under External Frequency Injection

Namdar Saniei, Aidin Tofangdarzade
Electrical and Computer Faculty
Shahid Beheshti University
Tehran, Iran
n-saniei@sbu.ac.ir, a.tofangdarzade@sbu.ac.ir

Wai Tung Ng
Department of Electrical and Computer Engineering
University of Toronto
Toronto, Ontario
ngwt@vrg.utoronto.ca

Abstract—In this work, time domain analysis is used to solve Adler’s equation to obtain the phase shift between a free-running oscillator and an externally injected oscillator. In addition, the time required to acquire locking between the two oscillators is also deduced. The dependency of the locking time versus characteristics of the oscillator such as the quality factor, Q, and its free-running frequency are also discussed. Finally, the analyses are verified by simulations of a typical LC oscillator under injection. The results of this analysis enable designers to evaluate the exact timing budget required to achieve injection locking.

Keywords: Adler equation, Injection locking, locking time, Oscillator

I. INTRODUCTION

When an autonomous oscillator is working with its free-running frequency, an external signal could be injected into the basic oscillator. If certain conditions are satisfied the injected frequency could take over the oscillation frequency of the oscillator. This process is called Injection Locking which has been widely used in modern analog electronics for many applications such as frequency division [1]-[3], multiplication [4], quantization noise reduction in synthesizers [5], and increasing free-running frequency of an oscillator [6]-[7].

There is a time interval, called locking time that the basic oscillator needs to give up its free-running frequency and begins to oscillate with the injected frequency which named Locking Time [8]. In some applications such as locking on pulsed signals the locking time becomes significant. Reference [8] defines this time in a one special case for high level injection1. Since resulted closed-form equation for phase is very unwieldy and complicating, numerical methods are used in this reference to depict some locking time related curves for this case that can be helpful just in this special case and no closed-form equation is proposed for locking time in this paper.

Phase domain graphical Phasor-Based analysis is presented in [9]-[11] which include basic concepts to fully describe the injection locking. Also other Time and Phase domain injection locking approaches have been introduced in other works; in [12]-[13] a model is introduced based on Impulse Sensitivity Function (ISF) for analysis of phase noise in injection locked oscillators. This approach is developed in [14] which employs ISF for describing injection locking in any type of oscillators. This phase-domain and complex model using ISF function is really suitable for frequency synchronization, and finding the suitable nodes for injection. It needs numerical methods for depicting ISF. So, it is not appropriate enough for design and extracting a closed-form equation for locking time but very well for optimization.

As another example, a delay based time-domain model for injection locking is introduced in [15] that is advantageous in any type of non-harmonic oscillators especially in the aspect of lock range, but not useful for concept of the locking time. Therefore, we need time- based simple procedure to obtain the Locking time as a function of time.

The underlying purpose of this research is to concentrate on the time domain aspects of this phenomenon, injection locking, and to extract all required parameters for its time domain analysis. As a final result, the time required for an oscillator to lock on to another one is obtained. This is useful in cases such as the determination on the overall time delay in a PLL loop.

In section II, injection locking is introduced. Mathematical relationships are arranged in a way that yields a primary and qualitative insight on injection locking. In sections III and IV, Adler’s equation will be solved first approximately, and then accurately, then, a new criterion will be introduced that can help us to obtain the time required to reach a stable state under an external frequency injection. In section V, the locking time derived in the previous section is verified using simulation results of a typical oscillator under injection.

\[ V_{in} \cos(\omega_{in} t) \]

\[ V_{osc} \cos(\omega_{osc} t + \theta) \]

Figure 1. Basic oscillator, output, and injected signals

1 Injection level is almost 0.9

Proceedings of ICEE 2010, May 11-13, 2010
978-1-4244-6761-7/10/$26.00 ©2010 IEEE
II. INJECTION LOCKING AND ADLER’S EQUATION

The main purpose of injection locking is to influence an oscillator with another. Suppose that we have an oscillator with a free-running frequency $\omega_0$ and quality factor of Q. If another oscillator with a frequency of $\omega_{\text{inj}}$ is injected in to the original oscillator, what will happen to the free running oscillator?

In order to operate properly, an oscillator needs to reach a frequency dependent phase shift with a value of $180^\circ$ in its inherent feedback loop. By applying an external signal to the oscillator, phase shift inside the feedback loop starts to change. In order to compensate this additional phase shift, the oscillator changes its instantaneous frequency $\omega(t)$. This frequency deviation continues until the instantaneous frequency from $\omega_{\text{inj}}$, at the start of injection, to the frequency $\omega_{\text{oc}}\cos(\omega_{\text{inj}} + \omega_{\text{L}})$. The instantaneous frequency of the output signal of oscillator changes to $\omega_{\text{oc}}\cos(\omega_{\text{inj}} + \omega_{\text{L}})$. This frequency approach is determined by Adler’s equation, as shown below:

$$\omega(t) = \frac{d}{dt}(\omega_{\text{inj}} t + \theta) = \omega_{\text{inj}} + \frac{d\theta}{dt} \quad (1)$$

where $\omega_{\text{inj}}$ has a specific and constant value. The angular frequency $\frac{d\theta}{dt}$ is the only term in (1), which can change the instantaneous frequency from $\omega_{\text{inj}}$ at the start of injection, to the final value; i.e. the injected frequency ($\omega_{\text{inj}}$). This important parameter ($\frac{d\theta}{dt}$) is characterized by Adler’s equation, as shown below:

$$\frac{d\theta}{dt} = \omega_{\text{inj}} - \omega_{\text{L}} \sin(\theta) = \frac{\omega_0 - \omega_{\text{inj}} - \omega_{\text{L}} \sin(\theta)}{2Q} \quad (2)$$

Where $\omega_{\text{L}}$ is the lock range. This means that the injected frequency can influence and acquire by the original oscillator if it lies in this range. In other words, if the distance between injected and primary frequency is smaller than $\omega_{\text{L}}$, the injected frequency falls within the lock range and injection locking can occur. Since, $\omega_{\text{inj}}$ may be greater or smaller than $\omega_{\text{inj}}$, the total lock range would, in fact, be $2\omega_{\text{L}}$, as shown in Fig. 2.

When $\frac{d\theta}{dt}$ goes to zero, locking will happen. Under this condition, equation (1) specifies that the instantaneous frequency becomes the same as the injected frequency. After locking has occurred, the compensating phase shift $\theta$, which has a non-zero value can be determined, as below:

$$\theta = \sin^{-1}\left(\frac{\omega_0 - \omega_{\text{inj}}}{\omega_{\text{L}}}\right) \quad (3)$$

As mentioned before, and shown in Fig. 2, in order to have successful locking, $\omega_{\text{inj}}$ must be within the lock range. Therefore, the minimum value of $\omega_{\text{inj}}$ is $\omega_0 - \omega_{\text{L}}$ and the maximum value of that is $\omega_0 + \omega_{\text{L}}$. Also, with these critical points, using (3), the variation of $\theta$ will be in the range:

$$-\pi/2 \leq \theta \leq \pi/2 \quad (4)$$

Using (1), the initial and final conditions of $\frac{d\theta}{dt}$ can be calculated, which can be helpful in solving the Adler’s differential equation.

For $(t=0)$ the instantaneous frequency is equal to the free-running frequency and after locking, to the $\omega_{\text{inj}}$ at $(t=t_{\text{lock}})$, therefore, the term $\omega_{\text{inj}}$ at $(t=t_{\text{lock}})$, where $t_{\text{lock}}$ is the time that the oscillator needs to reach its steady-state condition. Therefore, we can note the following relationship:

$$\omega(t = 0) = \omega_{\text{inj}} + \frac{d\theta}{dt} (t = 0) = \omega_0 \rightarrow \frac{d\theta}{dt} (t = 0) = \omega_0 - \omega_{\text{inj}} \quad (5)$$

$$\omega(t = t_{\text{lock}}) = \omega_{\text{inj}} + \frac{d\theta}{dt} (t = t_{\text{lock}}) = \omega_{\text{inj}} \rightarrow \frac{d\theta}{dt} (t = t_{\text{lock}}) = 0 \quad (6)$$

It is evident that in order to achieve locking we have to have a zero $\frac{d\theta}{dt}$. To get a better insight, let us re-write (2), in a more meaningful form, below:

$$\frac{d\theta}{dt} = (\omega_0 - \omega_{\text{L}} \sin(\theta)) - \omega_{\text{inj}} \quad (7)$$

As can be deduced from (7), there are two terms which start to compete with each other; when they become comparable the condition for locking is satisfied. The term $\omega_{\text{L}} \sin(\theta)$ is the only variable in $\frac{d\theta}{dt}$. Therefore, it is useful to express $\theta$ as an explicit function of time which is the focus of the following section.

III. TIME-DOMAIN APPROXIMATION OF ADLER’S EQUATION FOR SMALL $\theta$’S

In order not to get entangled in the mathematical aspects of the problem, we should begin with a simple analysis on the injection locking phenomenon, by assuming small $\theta$’s.

Based on (3), if $\omega_{\text{inj}}$ is close enough to $\omega_0$, $\sin(\theta)$ can be approximated as $\theta$. Therefore, (2) changes to the following form:

$$\frac{d\theta}{dt} = \omega_0 - \omega_{\text{inj}} - \theta \omega_{\text{L}} \quad (8)$$
Solving this differential equation with the initial condition \( [\theta = 0] \) as mentioned before, \( \theta \) can be obtained as a function of time, as seen in (9):

\[
\theta = \frac{\omega_0 - \omega_m}{\omega_L} (1 - e^{-\omega_L t}) \quad (9)
\]

Using this explicit time domain function, we take the derivative of \( \theta \) with respect to time and obtain \( \frac{d\theta}{dt} \) as a function of time. Using (1), the time domain function of \( \omega(t) \) can be extracted:

\[
\frac{d\theta}{dt} = (\omega - \omega_m) e^{-\omega_L t} \quad (10)
\]

\[
\omega(t) = \omega_m + (\omega_0 - \omega_m) e^{-\omega_L t} \quad (11)
\]

These equations are exactly in accordance with the findings in section II. As seen from (9), \( \theta \) is zero at \( t = 0 \) and goes to its final value \( (\omega_0 - \omega_m)/\omega_L \) after a long period of time (this is also consistent with (3), for small \( \theta \)). On the other hand, \( \frac{d\theta}{dt} \) has an initial value of \( (\omega_0 - \omega_m) \) and its final value approaches zero. Moreover, in this process, \( \omega(t) \) goes to \( \omega_m \) (from \( \omega_0 \)) as is expected. We can get a better picture of this process by re-writing (11), in a more meaningful form as below:

\[
\omega(t) = \omega_0 e^{-\omega_L t} + \omega_m (1 - e^{-\omega_L t}) \quad (12)
\]

This equation clearly shows that the oscillator gives up its free-running frequency, exponentially, while \( \omega_m \) dominates the value of \( \omega(t) \), as time goes by.

IV. EXACT TIME-DOMAIN SOLUTION OF ADLER’S EQUATION

In this section, we solve the Adler’s equation without approximation in a general form. In this analysis, essential parameters (e.g. \( \theta \), \( \omega \), \( \frac{d\theta}{dt} \)) are derived as functions of time. The injection locking phenomenon is also described mathematically in the time domain.

Using the variable substitution, \( u = \tan (\theta/2) \), and the trigonometric relationship below (13), Adler’s equation can be solved.

\[
\sin(\theta) = \tan(\theta/2)/(1 + \tan^2(\theta/2)) \quad (13)
\]

Using the same variable substitution mentioned above, we can derive:

\[
\frac{d\theta}{dt} = \frac{2du}{1 + u^2}, \quad \frac{d\theta}{dt} = \frac{2\frac{du}{dt}}{1 + u^2} \quad (14)
\]

To simplify the notations, the abbreviation \( (\Delta \omega_0 = \omega_0 - \omega_m) \) can be used. Adler’s equation can be re-written as (15):

\[
\frac{d\theta}{\Delta \omega_0 - \omega_L \sin \theta} = \frac{dt}{\frac{2du}{\Delta \omega_0}} = \frac{dt}{\frac{2\omega_m}{\Delta \omega_0} - u + 1} \quad (15)
\]

In order to solve this equation, the denominator has to be changed to a perfect square form:

\[
\begin{align*}
\Delta \omega_0 & = \omega_0 - \omega_L \sin \theta \sin(\theta/2)/(1 + \tan^2(\theta/2)) \\
& = \frac{2du}{1 + u^2} + 1 - \left(\frac{\omega_m}{\Delta \omega_0}\right)^2 \\
& = \frac{2du}{1 + u^2} + 1 - \left(\frac{\omega_m}{\Delta \omega_0}\right)^2 \quad (16)
\end{align*}
\]

On the sign of \( 1 - (\omega_m/\Delta \omega_0)^2 \) two solutions are possible. If the sign is positive, it means that \( \omega_m \) resides out of the lock range boundary. As evident from the following relationship, the oscillator will not lock to the injected signal [1-3], [9-10].

\[
1 - (\frac{\omega_m}{\Delta \omega_0})^2 < 0 \rightarrow \left| \frac{\omega_L}{\Delta \omega_0} \right| < 1 \rightarrow \left| \omega_L - \omega_m \right| < \left| \omega_0 - \omega_m \right| \quad (17)
\]

The focus of this work is on the complementary state, i.e., the term \( 1 - (\omega_m/\Delta \omega_0)^2 \) having a negative value. In this case injection locking can occur and the following relationship along with Fig. 3 clearly describe the situation:

\[
1 - (\frac{\omega_m}{\Delta \omega_0})^2 > 0 \rightarrow \left| \frac{\omega_L}{\Delta \omega_0} \right| > 1 \rightarrow \left| \omega_L - \omega_m \right| > \left| \omega_0 - \omega_m \right| \quad (18)
\]

In this case, the denominator of (16), can be separated in to two distinct roots, which yields an exponential solution. Thus, (16), can be re-written in the following form:
\[
\frac{2}{\Delta \omega_0} \frac{du}{dt} = \frac{d}{dt} \left( \frac{u - \omega_i - \Delta \omega_0}{\Delta \omega_0} \right) = \frac{2du/\Delta \omega_0}{(u-u_0)(u-u_2)} = dt
\]

Where,

\[
\begin{align*}
u_1 &= \frac{\omega_L}{\Delta \omega_0} + \frac{\sqrt{\omega_L^2 - \Delta \omega_0^2}}{\Delta \omega_0} \\
u_2 &= \frac{\omega_L}{\Delta \omega_0} - \frac{\sqrt{\omega_L^2 - \Delta \omega_0^2}}{\Delta \omega_0}
\end{align*}
\]

Once again, we use the initial condition \(0(t=0) = 0\) and manipulate the variable \(u\) as a function of \(t\) by solving (19), which in fact is the Adler’s equation.

\[
u = \tan(\frac{\theta}{2}) = \frac{1 - e^{-\Delta \omega_0 \theta / \omega_i}}{\omega_i + \frac{\Delta \omega_0}{\Delta \omega_0}} = \frac{1 - e^{-\Delta \omega_0 \theta / \omega_i}}{\omega_i + \frac{\Delta \omega_0}{\Delta \omega_0}}
\]

Where,

\[
\omega_i = \sqrt{\omega_L^2 - \Delta \omega_0^2} \quad \Delta \omega_0 = \begin{cases} +1, & \Delta \omega_0 > 0 \\ -1, & \Delta \omega_0 < 0 \end{cases}
\]

It is evident from the equations above that there are two solutions depending on the injected frequency being greater or smaller than the free-running frequency. First, we assume that \(\omega_{inj}\) is greater than \(\omega_0\) (positive \(\Delta \omega_0\)). With this assumption, (21) becomes

\[
u = \tan(\frac{\theta}{2}) = \frac{1 - e^{-\Delta \omega_0 \theta / \omega_i}}{\omega_i + \frac{\Delta \omega_0}{\Delta \omega_0}}
\]

For this case, the boundary condition can be calculated as below:

\[
t \to 0 \quad \Rightarrow \quad u \to 0 \quad \Rightarrow \quad \theta \to 0
\]

And

\[
t \to \infty \quad \Rightarrow \quad u \to \frac{\Delta \omega_0}{\omega_i + \omega_i} \Rightarrow \theta \to 2 \tan^{-1} \left( \frac{\Delta \omega_0}{\omega_i + \omega_i} \right)
\]

The second case is when \(\Delta \omega_0\) goes negative, corresponding to an injected frequency greater than \(\omega_0\). In a similar manner, the variable \(u\) can be evaluated as a function of time, using (21), (22).

\[
u = \tan(\frac{\theta}{2}) = \frac{1 - e^{-\Delta \omega_0 \theta / \omega_i}}{\omega_i + \frac{\Delta \omega_0}{\Delta \omega_0}}
\]

The boundary condition for this case is also equal to the ones found in (24), (25). It can be proven that the boundary condition for \(\theta\) found from (25) is equivalent to the one obtained from (3). This shows that the time domain representation of Adler’s equation completely matches his original equation. Since in the original Adler’s equation, the rate of change for \(\theta\) with respect to time \((d\theta/dt)\) is addressed, it is worth while to do the same using the new approach presented in this work. Due to symmetry, only negative \(\Delta \omega_0\) will be considered in the following analysis. Using (23) and (14), \(d\theta/dt\) can be expressed as a function of time:

\[
\frac{d\theta}{dt} = \frac{4 \Delta \omega_0 (\omega_i) \frac{e^{\Delta \omega_0 \theta / \omega_i}}{\omega_i}}{(\omega_i + \omega_i)(1 - e^{\Delta \omega_0 \theta / \omega_i})}
\]

Considering the exponential terms, both \(\theta\) and \(d\theta/dt\) start from an initial value and tend to their final constant values. In Fig. 4, typical curves of the abovementioned quantities are depicted, where their initial and final values are also shown.

In order to determine the time required for an oscillator to reach the steady state in post-injection situation, a 2% criterion within its maximum value is used.

\[
\frac{d\theta}{dt} (t = \frac{2}{100} \Delta \omega_0 \to t = t_{\text{Lock}})
\]

Using (27), it can be shown that the locking time can be calculated as:

\[
t_{\text{lock}} = \frac{1}{\Delta \omega_0} \ln \left[ \frac{400(\omega_i)^2 + 2\Delta \omega_0 (\omega_i + \omega_i)}{2(\omega_i + \omega_i)^2} \right]
\]
Using discussion presented above, the initial and locking conditions are obtained. They match exactly to the ones from the original Adler’s equation, expressed in (5), (6).

\[
\frac{d\theta}{dt}(t = 0) = \Delta \omega_0 = \omega_0 - \omega_{inj} \quad (30)
\]

\[
\frac{d\theta}{dt}(t = t_{lock}) = 0 \quad (31)
\]

Using this result, \( t_{lock} \) versus normalized frequency can be graphed for a variable parameter such as the quality factor (Q), as shown in Fig. 5.

This figure shows that an injection with a frequency equal to the free-running frequency results the minimum value of \( t_{lock} \). Therefore, when the injection frequency becomes more or less than the free-running frequency, \( t_{lock} \) becomes greater. Also, the extent of lock range is another parameter which affects the locking time directly. Assuming a constant \( \Delta \omega_0 \), the greater the lock range the shorter is the locking time.

V. SIMULATION RESULTS

A 3.65 GHz LC oscillator has been designed in a 0.18 μm CMOS technology. Fig. 6 shows the method of injection in the designed LC oscillator. The injected current level is approximately 1.62 dB below the oscillator current level and the lock range of this oscillator is 605 MHz which is equal to 0.166\( \omega_0 \). After 80 ns, the oscillator reaches an initial stable state. Then a signal is injected into the original oscillator with a frequency of 4.1 GHz. The simulated output result is shown in Fig. 7 on which the momentary variations in the output frequency can be observed. In order to measure the locking time from the simulations, a method similar to the phase detection in PLLs is used. The phase difference of the injected signal as the reference signal and the output is converted to amplitude variations via a phase detector. Now the rate of change in the resultant amplitude is measured at the instant the injected signal is applied (80 ns), and then is used to find the time matching the 2% criterion, or the locking time. The time measured this way is about 9.05 ns, while (27) predicts a locking time of 9.40 ns.

Also, the \( t_{lock} \) has been verified using a lower free-running oscillator with a frequency of 156 MHz and a lock range of 15.6 MHz. In this case the injected frequency is 160 MHz which results in a calculated and simulated locking time of 263 ns and 260 ns, respectively.
VI. CONCLUSION

Mathematical analysis of Adler’s equation for injection locking oscillators was performed. This time domain analysis, for the first time, revealed the essential parameters, such as the phase difference and the locking time, for oscillators under injection locking. Determination of the locking time is beneficial in the realistic estimation of the timing budget in circuits utilizing injection locking oscillators. A closed form solution for the locking time was presented, and its dependency on the characteristics of the oscillator was illustrated. The locking time relationship was verified using an LC oscillator, under injection, for a wide range of frequency.

ACKNOWLEDGMENT

The authors would like to thank Dr. A. Fotowat-Ahmadi from Sharif University for his thoughtful ideas, ITRC for their partial financial support of this project, and the ECE department of University of Toronto for their fruitful collaborations.

REFERENCES